

ON THE DISCRETE SPECTRUM OF QUANTUM LAYERS

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ABSTRACT. Consider a quantum particle trapped between a curved layer of constant width built over a complete, non-compact, \mathcal{C}^2 smooth surface embedded in \mathbb{R}^3 . We assume that the surface is asymptotically flat in the sense that the second fundamental form vanishes at infinity, and that the surface is not totally geodesic. This geometric setting is known as a *quantum layer*. We consider the quantum particle to be governed by the Dirichlet Laplacian as Hamiltonian. Our work concerns the existence of bound states with energy beneath the essential spectrum, which is mathematically equivalent to the existence of *discrete spectrum*. We first prove that if the Gauss curvature is integrable, and the surface is *weakly κ -parabolic*, then the discrete spectrum is non-empty. This result implies that if the total Gauss curvature is *non-positive*, then the discrete spectrum is non-empty. We next prove that if the Gauss curvature is *non-negative*, then the discrete spectrum is non-empty. Finally, we prove that if the surface is *parabolic*, then the discrete spectrum is non-empty if the layer is sufficiently thin.

1. INTRODUCTION

Mesoscopic physics describes length scales from one atom to micrometers. At these scales, the behavior of particles is no longer described by classical physics: quantum effects are observed. Numerous phenomena such as quantum dots and wells occur at the scale of mesoscopic physics, and *all nanotechnology is on the mesoscopic scale*.

Consider, for example, electrons trapped between two semi-conducting materials, or more generally, quantum particles trapped between hard walls. Mathematically, such situations are described using a *quantum layer*.

Let $p : \Sigma \rightarrow \mathbb{R}^3$ be an embedded surface in \mathbb{R}^3 . We will always make the following assumptions on Σ .

1.0.1. Hypotheses.

- (1) Σ is a \mathcal{C}^2 smooth surface;
- (2) Σ is orientable, complete, but non-compact;
- (3) Σ is not totally geodesic;

- (4) Σ is asymptotically flat in the sense that the second fundamental form, denoted B throughout this work, tends to zero at infinity.

A *quantum layer* over Σ is an oriented differentiable manifold $\Omega \cong \Sigma \times [-a, a]$ for some (small) positive number a . Let \vec{N} be the unit normal vector of Σ in \mathbb{R}^3 . Define

$$\tilde{p} : \Omega \rightarrow \mathbb{R}^3$$

by

$$\tilde{p}(x, t) = p(x) + t\vec{N}_x.$$

If a is small, then \tilde{p} is clearly an immersion. The Riemannian metric ds_Ω^2 is defined as the pull-back of the Euclidean metric via \tilde{p} . The Riemannian manifold (Ω, ds_Ω^2) is called the *quantum layer*. Physically, the quantum particles are trapped between two copies of the same semi-conducting material Σ at a uniform distance of $2a$ apart, where a is of mesoscopic scale. A natural question is:

Does there exist a geometric condition on Σ which guarantees the existence of bound states with energy beneath the essential spectrum?

Let us formulate this precisely. Let $\Delta = \Delta_\Omega$ be the Laplacian with respect to the Riemannian metric ds_Ω^2 , and assume the Dirichlet boundary condition. Since Ω is a smooth, complete manifold with boundary, the Dirichlet Laplacian is the Friedrichs extension of the Laplacian acting on $C_0^\infty(\Omega)$ and is self-adjoint. The spectrum of the Dirichlet Laplacian consists of two parts: discrete, isolated eigenvalues of finite multiplicity and essential spectrum. We distinguish the eigenvalues which are *disjoint* from the essential spectrum and refer to these as the *discrete spectrum*, since there may also be embedded eigenvalues within the essential spectrum. Physically, the quantum particles are governed by the Dirichlet Laplacian as Hamiltonian, and eigenvalues correspond to the Dirichlet energy of bound states. Therefore:

The existence of discrete spectrum is equivalent to the existence of bound states in the physical model whose energy is beneath the essential spectrum.

Let κ be the Gauss curvature of Σ throughout this paper. Our work is motivated by the following conjecture.

Conjecture 1. *Under the preceding assumptions (1.0.1) on Σ , if*

$$(1.1) \quad \int_{\Sigma} |\kappa| d\Sigma < +\infty,$$

then there exists an $\alpha = \alpha(\Sigma)$ such that for all $a \in (0, \alpha)$, the discrete spectrum of the quantum layer over Σ of width $2a$ is non-empty.

Remark 1.1. By a theorem of Huber [5], if (1.1) is valid, then Σ is conformal to a compact Riemann surface with finitely many points removed. Moreover, White [12] proved that if

$$(1.2) \quad \int_{\Sigma} \kappa^- d\Sigma < +\infty,$$

where

$$\kappa = \begin{cases} \kappa^+ & \kappa \geq 0 \\ -\kappa^- & \kappa < 0 \end{cases},$$

then

$$\int_{\Sigma} |\kappa| d\Sigma < +\infty.$$

Thus (1.1) can be weakened to (1.2).

Conjecture 1 was proven under the condition

$$(1.3) \quad \int_{\Sigma} \kappa d\Sigma \leq 0$$

through the work of Duclos, Exner and Krejčířík [3] and Carron, Exner, and Krejčířík [2]. Moreover, [2] also proved that Conjecture 1 holds if the gradient of the mean curvature is locally square integrable, and the total mean curvature is infinite.

Our work focuses on the remaining case:

$$\int_{\Sigma} \kappa d\Sigma > 0.$$

By the theorem of Huber [5], there exists a compact Riemann surface $\overline{\Sigma}$ and finitely many points p_1, \dots, p_s such that

$$\Sigma = \overline{\Sigma} \setminus \{p_1, \dots, p_s\}.$$

In particular, Σ has finitely many ends, E_i ($1 \leq i \leq s$). By a theorem of Hartman [4, Theorem 6.1, 7.1], we have

$$(1.4) \quad \int_{\Sigma} \kappa d\Sigma = 2\pi \left(\chi(\Sigma) - \sum_{i=1}^s \lambda_i \right)$$

where $\chi(\Sigma)$ is the Euler characteristic number of Σ , and λ_i are the isoperimetric constants at each end E_i defined by

$$\lambda_i = \lim_{r \rightarrow \infty} \frac{\text{vol}(B(r) \cap E_i)}{\pi r^2}.$$

The existence of these limits follows from the integrability of the Gauss curvature (1.1).

We also have

$$\chi(\Sigma) \leq \chi(\overline{\Sigma}) - s = 2 - 2g(\overline{\Sigma}) - s < 2.$$

This together with (1.4) implies that $\chi(\Sigma) = 1$, and hence the surface is diffeomorphic to \mathbb{R}^2 . Consequently,

$$(1.5) \quad 0 < \int_{\Sigma} \kappa d\Sigma \leq 2\pi.$$

Although the topology of the surface is completely known, this is the only remaining case in which the conjecture has not yet been proven.

We recall the main results of [2, 3].

Theorem 1.1 (Duclos, Exner and Krejčířík). *Let Σ be a \mathcal{C}^2 -smooth complete simply connected non-compact surface with a pole embedded in \mathbb{R}^3 . Let the layer $\Omega \cong \Sigma \times [-a, a]$ built over the surface be not self-intersecting. If the surface is not a plane, but it is asymptotically flat, then if Σ satisfies condition (1.6) below, each of the following implies Conjecture 1.*

- (1) *The Gauss curvature satisfies (1.1) and (1.3);*
- (2) *Σ is \mathcal{C}^3 smooth, and a is sufficiently small;*
- (3) *Σ is \mathcal{C}^3 smooth, the Gauss curvature is integrable, the gradient of the mean curvature $\nabla_g H$ is L^2 integrable, and the total mean curvature is infinite;*
- (4) *the Gauss curvature is integrable and Σ is cylindrically symmetric.*

In [2], Carron, Exner and Krejčířík proved that the conjecture holds under more general conditions. They no longer required the surface to have a pole, and they also proved Conjecture 1 under the additional assumptions that the gradient of the mean curvature is square integrable and the total mean curvature is infinite.

Theorem 1.2 (Carron, Exner and Krejčířík). *Let Σ be a complete asymptotically flat, noncompact connected surface of class \mathcal{C}^2 embedded in \mathbb{R}^3 and such that the Gauss curvature satisfies (1.1). Let the layer Ω of width $2a$ be defined so that Ω does not overlap, and Σ satisfies condition (1.6) below. Then, any of the following imply Conjecture 1.*

- (1) *The Gauss curvature satisfies (1.3);*
- (2) *a is small enough, and the gradient of the mean curvature $\nabla_g H$ is locally L^2 integrable;*
- (3) *the gradient of the mean curvature $\nabla_g H$ is L^2 integrable, and the total mean curvature is infinite;*
- (4) *Σ contains a cylindrically symmetric end with a positive total Gauss curvature.*

The general method used in both [2, 3] is: first compute the infimum of the essential spectrum, next construct appropriate test functions, and finally apply the variational principle to prove that if one of the conditions is satisfied, then there must be an eigenvalue strictly less than the essential spectrum. The pole and symmetry assumption (4) were necessary in [3] because their test functions are radially symmetric.

The first main result of the present paper generalizes [2, 3] by demonstrating that Conjecture 1 holds if the surface is *weakly κ -parabolic*. (We refer to § 3 for the definition of weak κ -parabolicity.)

Theorem 1.3. *Let Σ be a complete surface in \mathbb{R}^3 which satisfies the hypotheses (1.0.1), and assume that Σ is weakly κ -parabolic. Then, there exists $\alpha > 0$, depending only on the supremum of the norm of the second fundamental form, such that for all $a \in (0, \alpha)$, the discrete spectrum of the quantum layer over Σ of width $2a$ is non-empty.*

Although the proof of Theorem 1.3 is based on the same principles used in [2, 3], our theorem not only generalizes their results, but also shows that their argument fits nicely into the notion of weak κ -parabolicity, which provides a geometric abstraction of their argument.

The class of layers considered in [2, 3] was already quite broad, but not exhaustive. For example, a question raised in [3] which remained unanswered in [2] is:

Question 1.1. Does Conjecture 1 also hold for thick layers¹ built over surfaces of strictly positive total Gauss curvature without assuming cylindrical symmetry or square-integrability of the gradient of the mean curvature?

The first author and C. Lin proved in [9, Theorem 1.1] that Conjecture 1 holds when Σ can be represented as the graph of a convex function satisfying certain conditions. Our next theorem generalizes [9] and gives an affirmative answer to the above question under the additional assumption that the Gauss curvature is everywhere non-negative.

Theorem 1.4. *Let Σ be a complete surface in \mathbb{R}^3 which satisfies the hypotheses (1.0.1). Assume that the Gauss curvature of Σ is non-negative and satisfies (1.1). Then for all a such that*

$$(1.6) \quad a \in (0, B_\infty^{-1}), \quad B_\infty := \sup_{p \in \Sigma} \|B(p)\|,$$

¹So-called “thick layers” are those whose thickness satisfies (1.6) and is not otherwise restricted.

the discrete spectrum of the quantum layer over Σ of width $2a$ is non-empty.

Remark 1.2. The condition (1.6) is merely a technicality to ensure that Ω is immersed in \mathbb{R}^3 and is slightly weaker than the non-overlapping assumption made in [2, 3].

The proof of Theorem 1.4 is more subtle. Test functions similar to those used in [2, 3] rely on the weak κ -parabolicity of the surfaces, but a surface with non-negative Gauss curvature will *not* be weakly κ -parabolic. The main idea is to work on annuli, rather than on disks. In general, the integration of the mean curvature outside a compact set may be quite small since the surface is asymptotically flat. But using a result of White [12], we actually know that the total mean curvature is at least of linear growth. This estimate plays a crucial role in the proof.

In fact, our final main result is more general than Conjecture 1. Any surface whose Gauss curvature is integrable must be *parabolic* (see §3 for the definition), yet not all parabolic surfaces have integrable Gauss curvature, as demonstrated in §5.

Theorem 1.5. *Let Σ be a complete, parabolic surface in \mathbb{R}^3 which satisfies the hypotheses (1.0.1). Then, there exists $\alpha > 0$ such that the discrete spectrum of the quantum layer over Σ of width $2a$ is non-empty for all $a \in (0, \alpha)$.*

This note is organized as follows. In section § 2, we recall the variational principles for the essential spectrum and the ground state, and we determine the infimum of the essential spectrum. In § 3, we introduce the notion of κ -parabolicity and prove Theorem 1.3. The proof of Theorem 1.4 comprises § 4. We conclude in § 5 with the proof of Theorem 1.5 and a discussion of further generalizations.

2. VARIATIONAL PRINCIPLES AND THE INFIMUM OF THE ESSENTIAL SPECTRUM

It is well known that

$$(2.1) \quad \sigma_0 = \inf_{f \in C_0^\infty(\Omega)} \frac{\int_\Omega |\nabla f|^2 d\Omega}{\int_\Omega f^2 d\Omega}$$

is the infimum of the Laplacian.

For a compact set $E \subset \Sigma$, we shall use the notation

$$f \in \mathcal{C}_0^\infty(\Sigma \setminus E)$$

to denote a function $f \in \mathcal{C}_0^\infty(\Sigma)$ whose support lies in $\Sigma \setminus E$. The infimum of the essential spectrum is

$$(2.2) \quad \sigma_{ess} = \sup_K \inf_{f \in C_0^\infty(\Omega \setminus K)} \frac{\int_\Omega |\nabla f|^2 d\Omega}{\int_\Omega f^2 d\Omega}$$

where K runs over all compact subsets of Ω . Since $\Omega = \Sigma \times [-a, a]$, it is not hard to see that

$$(2.3) \quad \sigma_{ess} = \sup_{K \subset \Sigma} \inf_{f \in C_0^\infty(\Omega \setminus K \times [-a, a])} \frac{\int_\Omega |\nabla f|^2 d\Omega}{\int_\Omega f^2 d\Omega},$$

where K runs over all compact subsets of Σ .

It follows that $\sigma_0 \leq \sigma_{ess}$. Moreover, we have the following well-known result.

Proposition 2.1. *If $\sigma_0 < \sigma_{ess}$, then the discrete spectrum is non-empty.*

□

Let (x_1, x_2) be a local coordinate system of Σ . Then (x_1, x_2, t) defines a local coordinate system of Ω . Such a local coordinate system is called a *Fermi coordinate system*. Let $x_3 = t$, and let $ds_\Omega^2 = G_{ij} dx_i dx_j$. Then we have

$$(2.4) \quad G_{ij} = \begin{cases} (p + t\vec{N})_{x_i} (p + t\vec{N})_{x_j} & 1 \leq i, j \leq 2; \\ 0 & i = 3, \text{ or } j = 3, \text{ but } i \neq j; \\ 1 & i = j = 3. \end{cases}$$

We will demonstrate below that the infimum of the essential spectrum is equal to the spectral threshold of the planar quantum layer of width $2a$, namely, $\pi^2/4a^2$.

We make the following definitions. For a smooth function f on Ω , let

$$(2.5) \quad Q(f, f) = \int_\Omega |\nabla f|^2 d\Omega - \frac{\pi^2}{4a^2} \int_\Omega f^2 d\Omega;$$

$$(2.6) \quad Q_1(f, f) = \int_\Omega |\nabla' f|^2 d\Omega;$$

$$(2.7) \quad Q_2(f, f) = \int_\Omega \left(\frac{\partial f}{\partial t} \right)^2 d\Omega - \frac{\pi^2}{4a^2} \int_\Omega f^2 d\Omega,$$

where

$$|\nabla' f|^2 = \sum_{i,j=1}^2 G^{ij} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j}$$

is the square of the norm of the horizontal differential.

Obviously, we have

$$Q(f, f) = Q_1(f, f) + Q_2(f, f),$$

and

$$\int_{\Omega} |\nabla f|^2 d\Omega = \int_{\Omega} |\nabla' f|^2 d\Omega + \int_{\Omega} \left(\frac{\partial f}{\partial t} \right)^2 d\Omega.$$

Clearly, we have

$$\int_{\Omega} |\nabla f|^2 d\Omega \geq \int_{\Omega} \left(\frac{\partial f}{\partial t} \right)^2 d\Omega.$$

Let $ds_{\Sigma}^2 = g_{ij}dx_i dx_j$ be the Riemannian metric of Σ with respect to the coordinates (x_1, x_2) . We shall compare the matrices $(G_{ij})_{1 \leq i, j \leq 2}$ and (g_{ij}) outside a big compact set of Σ . By (2.4), we have

$$(2.8) \quad G_{ij} = g_{ij} + tp_{x_i} \vec{N}_{x_j} + tp_{x_j} \vec{N}_{x_i} + t^2 \vec{N}_{x_i} \vec{N}_{x_j},$$

where we note that $g_{ij} = p_i p_j$.

Using (2.8), we have

$$(2.9) \quad \det(G_{ij}) = \det(g_{ij})(1 - Ht + \kappa t^2).$$

The mean curvature H is defined here to be the trace of the second fundamental form.

We assume that at the point x , local coordinates (x_1, x_2) are chosen such that $g_{ij} = \delta_{ij}$. We have the estimate

$$(2.10) \quad |G_{ij} - \delta_{ij}| \leq a \|B\|,$$

where B is the second fundamental form of the surface Σ . Based on these calculations, we have the following.

Proposition 2.2. *Let Σ be an embedded surface in \mathbb{R}^3 which satisfies hypotheses (1.0.1). Then, for any a which satisfies (1.6), the quantum layer $\Omega \cong \Sigma \times [-a, a]$ is an immersed submanifold of \mathbb{R}^3 . Moreover, for any $\varepsilon > 0$, there is a compact set K of Σ such that on $\Sigma \setminus K$ we have*

$$(2.11) \quad (1 - \varepsilon) \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \leq \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \leq (1 + \varepsilon) \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}.$$

In particular, we have

$$(2.12) \quad (1 - \varepsilon)^2 d\Sigma dt \leq d\Omega \leq (1 + \varepsilon)^2 d\Sigma dt.$$

On the other hand, there exists $\alpha = \alpha(\Sigma, \varepsilon) > 0$ such that for all $a \in (0, \alpha)$, the above inequalities hold at any point of Σ .

Proof. It follows from (2.9) and (2.10) that Ω is an immersed submanifold whenever a satisfies (1.6). By (2.10) and the assumption that Σ is asymptotically flat, for any $\varepsilon > 0$, there exists a compact subset $K \subset \Sigma$ such that (2.11) holds on $\Sigma \setminus K$. Since $\|B\|$ vanishes at infinity, both H and κ also vanish at infinity. Therefore, (2.12) follows from (2.9). Finally, for a fixed $\varepsilon > 0$, we may choose a compact subset $K \subset \Sigma$ such that both (2.11) and (2.12) hold on $\Sigma \setminus K$, and since K is compact, by (2.9) and (2.10), we may choose a sufficiently small such that (2.11) and (2.12) hold also on K . \square

The following technical lemma shall be used throughout the remaining sections.

Lemma 2.1. *For a surface Σ which satisfies (1.0.1) and (1.1), let j be a smooth function on Σ , and let*

$$\chi(t) = \cos\left(\frac{\pi t}{2a}\right).$$

Then there exist universal constants C_1 and C_2 which depend only on B_∞ (the supremum of the norm of the second fundamental form) such that

$$Q(j\chi t, j\chi t) \leq C_1 a \int_{\Sigma} j^2 d\Sigma + C_2 a^3 \int_{\Sigma} |\nabla j|^2 d\Sigma.$$

Proof. It is a straightforward exercise to compute:

$$\begin{aligned}
 \int_{-a}^a \chi(t)^2 dt &= a \\
 \int_{-a}^a \chi(t)^2 t^2 dt &= \frac{a^3(\pi^2 - 6)}{3\pi^2}, \\
 \int_{-a}^a \chi(t)^2 t^4 dt &= \frac{a^5(120 - 20\pi^2 + \pi^4)}{5\pi^4}, \\
 \int_{-a}^a (\chi'(t))^2 t^2 dt &= \frac{a(6 + \pi^2)}{12}, \\
 \int_{-a}^a (\chi'(t))^2 t^4 dt &= \frac{a^3(20\pi^2 - 120 + \pi^4)}{20\pi^2}, \\
 \int_{-a}^a \chi'(t)\chi(t)t dt &= -a, \\
 \int_{-a}^a \chi'(t)\chi(t)t^3 dt &= \frac{a^3(6 - \pi^2)}{\pi^2}.
 \end{aligned}
 \tag{2.13}$$

By (2.9), (2.8), and since j is independent of t , there is a constant C_0 which depends only on B_∞ such that

$$Q_1(j\chi t, j\chi t) \leq C_0 \int_{-a}^a \int_{\Sigma} |\nabla j|^2 \chi^2 t^2 (1 + Ht + \kappa t^2) d\Sigma dt.$$

Since $\chi^2 t^2$ is an even function

$$Q_1(j\chi t, j\chi t) \leq C_0 \int_{-a}^a \int_{\Sigma} |\nabla j|^2 \chi^2 t^2 (1 + \kappa t^2) d\Sigma dt.$$

Therefore, by the above calculations

$$Q_1(j\chi t, j\chi t) \leq C_0 \left(\frac{a^3(\pi^2 - 6)}{3\pi^2} \int_{\Sigma} |\nabla j|^2 d\Sigma + \frac{a^5(120 - 20\pi^2 + \pi^4)}{5\pi^4} \int_{\Sigma} |\nabla j|^2 \kappa d\Sigma \right).$$

Let

$$C_2 := C_0 \left(\frac{\pi^2 - 6}{3\pi^2} + \frac{a^2(120 - 20\pi^2 + \pi^4)}{5\pi^4} B_\infty \right).$$

Next, we have

$$Q_2(j\chi t, j\chi t) = \int_{-a}^a \int_{\Sigma} ((\chi')^2 t^2 + \chi^2 + 2\chi\chi' t) (1 + Ht + \kappa t^2) j^2 d\Sigma dt.$$

Since $(\chi')^2 t^2 + \chi^2 + 2\chi\chi' t$ is an even function, we have

$$Q_2(j\chi t, j\chi t) = \int_{-a}^a \int_{\Sigma} ((\chi')^2 t^2 + \chi^2 + 2\chi\chi' t) (1 + \kappa t^2) j^2 d\Sigma dt.$$

By the above calculations and the definition of Q_2 , we compute

$$Q_2(j\chi t, j\chi t) = a \int_{\Sigma} j^2 d\Sigma + \frac{a^3(\pi^2 - 6)}{3\pi^2} \int_{\Sigma} j^2 \kappa d\Sigma.$$

Letting

$$C_1 := 1 + \frac{a^2(\pi^2 - 6)}{3\pi^2} B_\infty,$$

since $Q = Q_1 + Q_2$, we have

$$Q(j\chi t, j\chi t) \leq C_1 a \int_{\Sigma} j^2 d\Sigma + C_2 a^3 \int_{\Sigma} |\nabla j|^2 d\Sigma.$$

□

Based on the preceding results and the variational principle, we are able to determine σ_{ess} . The following lemma is originally due to [2, 3], but we include a short proof for completeness.

Lemma 2.2. *Let Σ be an embedded surface in \mathbb{R}^3 which satisfies the hypotheses (1.0.1), and assume the Gauss curvature satisfies (1.1). Then, for any quantum layer Ω built over Σ of width $2a > 0$, where a satisfies (1.6),*

$$\sigma_{ess} = \frac{\pi^2}{4a^2}.$$

Proof. We first prove that

$$\sigma_{ess} \geq \frac{\pi^2}{4a^2}.$$

Let $\varepsilon > 0$ be given, and let K be a compact set of Σ as in Proposition 2.2. Let $\tilde{K} \subset \Omega$ be defined as

$$\tilde{K} \cong K \times [-a, a].$$

For $f \in C_0^\infty(\Omega \setminus \tilde{K})$, by Proposition 2.2,

$$\int_{\Omega} \left(\frac{\partial f}{\partial t} \right)^2 d\Omega \geq (1-\varepsilon)^2 \int_{\Sigma} \int_{-a}^a \left(\frac{\partial f}{\partial t} \right)^2 dt d\Sigma \geq (1-\varepsilon)^2 \frac{\pi^2}{4a^2} \int_{\Sigma} \int_{-a}^a f^2 dt d\Sigma,$$

where the last inequality follows from the 1-dimensional Poincaré inequality. By Proposition 2.2 again, we have

$$\int_{\Omega} |\nabla f|^2 d\Omega \geq \frac{(1-\varepsilon)^2 \pi^2}{(1+\varepsilon)^2 4a^2} \int_{\Omega} f^2 d\Omega,$$

which by the variational principle for σ_{ess} implies

$$\sigma_{ess} \geq \frac{(1-\varepsilon)^2 \pi^2}{(1+\varepsilon)^2 4a^2}.$$

Letting $\varepsilon \rightarrow 0$ completes the proof of the first inequality.

To complete the proof of the lemma, we demonstrate the estimate

$$(2.14) \quad \sigma_{ess} \leq \frac{\pi^2}{4a^2}.$$

Since the Gauss curvature tends to zero, it is well known (see [1]) that the infimum of the essential spectrum of Σ is zero. Therefore, for any compact set K and any $\varepsilon > 0$, there exists a smooth function $\varphi \in C_0^\infty(\Sigma \setminus K)$ such that

$$\int_{\Sigma} |\Delta \varphi|^2 d\Sigma \leq \varepsilon \int_{\Sigma} \varphi^2 d\Sigma.$$

It follows that

$$\int_{\Sigma} |\nabla \varphi|^2 d\Sigma = - \int_{\Sigma} \varphi \Delta \varphi d\Sigma \leq \sqrt{\int_{\Sigma} \varphi^2 d\Sigma} \cdot \sqrt{\int_{\Sigma} (\Delta \varphi)^2 d\Sigma} \leq \sqrt{\varepsilon} \int_{\Sigma} \varphi^2 d\Sigma.$$

Using Proposition 2.2, for sufficiently large K , we have

$$\int_{\Omega} |\nabla' \varphi|^2 d\Omega \leq 2(1 + \varepsilon)a \int_{\Sigma} |\nabla \varphi|^2 d\Sigma \leq 2\varepsilon(1 + \varepsilon)a \int_{\Sigma} \varphi^2 d\Sigma.$$

As in Lemma 2.1, we let

$$\chi(t) = \cos(\pi t/2a)$$

and consider the function $\varphi\chi$ on Ω . Thus by (2.13) and (2.9), since χ^2 and $(\chi')^2$ are even functions, we have

$$\begin{aligned} Q_2(\varphi\chi, \varphi\chi) &= \int_{\Sigma} \varphi^2 d\Sigma \int_{-a}^a \left((\chi'(t))^2 - \frac{\pi^2}{4a^2} \chi(t)^2 \right) dt \\ &\quad + \int_{\Sigma} \varphi^2 \kappa d\Sigma \int_{-a}^a t^2 \left((\chi'(t))^2 - \frac{\pi^2}{4a^2} \chi(t)^2 \right) dt. \end{aligned}$$

By (2.13) we have

$$(2.15) \quad Q_2(\varphi\chi, \varphi\chi) = a \int_{\Sigma} \varphi^2 \kappa d\Sigma,$$

and hence

$$Q(\varphi\chi, \varphi\chi) \leq 2a\varepsilon(1 + \varepsilon) \int_{\Sigma} \varphi^2 d\Sigma + a \int_{\Sigma} \varphi^2 \kappa d\Sigma.$$

By Proposition 2.2 and (1.1), there exists an ε_1 sufficiently small such that

$$\int_{\Omega} (\varphi\chi)^2 d\Omega \geq 2(1 - \varepsilon_1)a \int_{\Sigma} \varphi^2 d\Sigma,$$

and

$$\int_{\Sigma} \varphi^2 \kappa d\Sigma \leq \varepsilon_1 \int_{\Sigma} \varphi^2 d\Sigma.$$

By the variational principle for σ_{ess} and the definition of Q , we have

$$\sigma_{ess} - \frac{\pi^2}{4a^2} \leq \frac{Q(\varphi\chi, \varphi\chi)}{\int_{\Omega} (\varphi\chi)^2 d\Omega} \leq \frac{2\varepsilon(1 + \varepsilon) + \varepsilon_1}{2(1 - \varepsilon_1)}$$

This proves the lemma. \square

3. κ -PARABOLICITY

We refer to [7] for the following definition and basic properties of parabolic manifolds.

Definition 3.1. *A complete manifold is parabolic if it does not admit a positive Green's function. Otherwise it is non-parabolic.*

We first establish the following well-known result.

Proposition 3.1. *Assume that $\kappa \in L^1(\Sigma)$. Then there exists a positive constant c_1 such that the volume growth of Σ satisfies*

$$\int_{B(R)} d\Sigma < c_1 R^2.$$

Proof. By the results of Huber [5] and Hartman [4], Σ has only finitely many ends, $\{E_i\}_{i=1}^s$ and at each end,

$$\lambda_i = \lim_{r \rightarrow \infty} \frac{\text{vol}(B(r) \cap E_i)}{\pi r^2}$$

exists. This proves the proposition. □

By the above proposition, a surface whose Gauss curvature $\kappa \in L^1$ is a parabolic manifold (cf. [7]).

The definition of parabolicity is equivalent to: *the capacity of any ball of radius R is zero*. That is, for any positive R and ε , there exists a smooth function φ which satisfies

$$\begin{aligned} \varphi &\in C_0^\infty(\Sigma); \\ \varphi &\equiv 1 \text{ on } B(R); \\ 0 &\leq \varphi \leq 1; \\ \int_{\Sigma} |\nabla \varphi|^2 d\Sigma &< \varepsilon. \end{aligned}$$

For the rest of the paper, we shall repeatedly use the above equivalent capacity definition of parabolicity.

Parallel to the above, we make the following definition of weak κ -parabolicity.

Definition 3.2. *The κ -capacity of a subset $E \subset \Sigma$ is defined to be the infimum of*

$$\int_{\Sigma} (|\nabla \varphi|^2 + \kappa \varphi^2) d\Sigma,$$

where φ is a smooth function on Σ with compact support, and $\varphi \equiv 1$ in a neighborhood of E .

We say Σ is weakly κ -parabolic, if either Σ is a minimal surface, or there exists $p \in \Sigma$ such that $H(p) \neq 0$, and the κ -capacity of a neighborhood of p is non-positive.

Proof of Theorem 1.3. If Σ is a minimal surface, then $\kappa \leq 0$. Consequently the total Gauss curvature is nonpositive, and the theorem follows from the results in [2, 3].

By Lemma 2.2 and the variational principles, it suffices to prove

$$\sigma_0 < \sigma_{ess}$$

for the remaining cases.

Let $p \in \Sigma$ be a point such that $H(p) \neq 0$. We assume that there is a constant $\varepsilon_1 > 0$ such that $|H(p)| > \varepsilon_1$ on the ball of radius δ centered at p , which is denoted $B_p(\delta)$, with δ a fixed positive constant. For any $\varepsilon_2 > 0$, by the definition of weakly κ -parabolic (and choosing $\delta > 0$ smaller if necessary), there exists a smooth function φ with compact support such that $\varphi \equiv 1$ on $B_p(\delta)$, and

$$(3.1) \quad \int_{\Sigma} (|\nabla \varphi|^2 + \kappa \varphi^2) d\Sigma < \varepsilon_2.$$

Let j be a smooth function such that the support of j is contained in $B_p(\delta)$, and

$$\left| \int_{\Sigma} H j d\Sigma \right| > \varepsilon_3 > 0$$

for some positive constant $\varepsilon_3 > 0$. For a suitable choice of orientation, we may assume that

$$\int_{\Sigma} H j d\Sigma > \varepsilon_3 > 0.$$

By (2.15), we have

$$Q_2(\varphi\chi, \varphi\chi) = a \int_{\Sigma} \varphi^2 \kappa d\Sigma.$$

By Proposition 2.2, for any $\varepsilon_4 > 0$, there exists $\alpha = \alpha(\varepsilon_4) > 0$ such that for all $a \in (0, \alpha)$,

$$Q_1(\varphi\chi, \varphi\chi) = \int_{\Omega} |\nabla' \varphi|^2 \chi^2 d\Omega \leq (1 + \varepsilon_4) a \int_{\Sigma} |\nabla \varphi|^2 d\Sigma.$$

Since $Q = Q_1 + Q_2$, and since the inequality (3.1) is *strict*, using (2.15), we can choose ε_4 sufficiently small and a corresponding α such that

$$Q(\varphi\chi, \varphi\chi) \leq \int_{\Sigma} (a(1 + \varepsilon_4)|\nabla \varphi|^2 + a\kappa\varphi^2) d\Sigma \leq a\varepsilon_2.$$

Since j assumes values in $[0, 1]$ and depends only on the choice of δ which is *fixed*, by Lemma 2.1 there is a constant c_1 such that

$$Q(j\chi(t)t, j\chi(t)t) \leq c_1 a.$$

The support of j is contained in $\{\varphi \equiv 1\}$, so by (2.6) we have

$$Q_1(\varphi\chi(t), j\chi(t)t) = 0.$$

By (2.9), and since $\varphi \equiv 1$ on the support of j , we have

$$Q_2(\varphi\chi(t), j\chi(t)t) = \int_{\Omega} \chi'(t)(\chi(t)t)' j d\Omega - \frac{\pi^2}{4a^2} \int_{\Omega} j\chi^2(t)t d\Omega.$$

We compute that

$$(3.2) \quad \chi'(t)(\chi(t)t)' - \frac{\pi^2}{4a^2} \chi^2(t)t = -\frac{\pi}{4a} \sin\left(\frac{\pi t}{a}\right) - \frac{\pi^2 t}{4a^2} \cos\left(\frac{\pi t}{a}\right)$$

is an odd function, and thus

$$Q_2(\varphi\chi(t), j\chi(t)t) = - \int_{\Sigma} H j d\Sigma \cdot \int_{-a}^a (\chi'(t)(\chi(t)t)' - \frac{\pi^2}{4a^2} \chi^2(t)t) dt.$$

But

$$(3.3) \quad \int_{-a}^a (\chi'(t)(\chi(t)t)' - \frac{\pi^2}{4a^2} \chi^2(t)t) dt = - \int_{-a}^a \chi'(t)\chi(t)t dt = \frac{a}{2}$$

by a straightforward computation. Therefore,

$$(3.4) \quad Q(\varphi\chi(t), j\chi(t)t) = -\frac{a}{2} \int_{\Sigma} H j d\Sigma.$$

Let $\varepsilon > 0$. Using all the above estimates, we have

$$Q(\varphi\chi(t) + \varepsilon j\chi(t)t, \varphi\chi(t) + \varepsilon j\chi(t)t) < a\varepsilon_2 - a\varepsilon_3\varepsilon + c_1 a\varepsilon^2.$$

Since ε_2 can be made arbitrary small, and $\varepsilon_3 > 0$ is fixed, for a suitable choice of ε , we have

$$Q(\varphi\chi(t) + \varepsilon j\chi(t)t, \varphi\chi(t) + \varepsilon j\chi(t)t) < 0,$$

for all $a \in (0, \alpha)$, for α chosen sufficiently small. The conjecture then follows from the definition of Q , the variational principle and Lemma 2.2. \square

The following sufficient condition for κ -parabolicity implies that Theorem 1.3 is indeed a generalization of both [2, 3].

Lemma 3.1. *Let Σ be a complete surface such that*

$$\int_{\Sigma} |\kappa| d\Sigma < \infty,$$

and

$$(3.5) \quad \int_{\Sigma} \kappa d\Sigma \leq 0.$$

Then Σ is weakly κ -parabolic.

Proof. First, if the mean curvature $H \equiv 0$ on Σ , then Σ is a minimal surface, and there is nothing to prove. So, we assume there exists some $p \in \Sigma$ such that $H(p) \neq 0$. Since κ is integrable, Σ is parabolic. That is, for any ball $B(R)$ of radius R centered at p , there exists a smooth function $0 \leq \varphi \leq 1$ with compact support such that $\varphi \equiv 1$ on $B(R)$ and

$$\int_{\Sigma} |\nabla \varphi|^2 d\Sigma < \varepsilon,$$

On the other hand, by the integrability of κ , for sufficiently large R ,

$$\int_{\Sigma \setminus B(R)} \kappa \varphi^2 d\Sigma < \varepsilon, \quad \int_{\Sigma \setminus B(R)} |\kappa| d\Sigma < \varepsilon,$$

and therefore, by the assumption (3.5),

$$\int_{\Sigma} \kappa \varphi^2 d\Sigma < 2\varepsilon.$$

It follows that for any $\varepsilon > 0$ and R sufficiently large, there exists a function $\varphi \in C_0^\infty(\Sigma)$ such that $\varphi \equiv 1$ on $B(R)$, and

$$\int_{\Sigma} (|\nabla \varphi|^2 + \kappa \varphi^2) d\Sigma < 3\varepsilon.$$

This estimate proves the lemma. \square

Remark 3.1. There are many examples of surfaces which are weakly κ parabolic but whose total Gauss curvature is positive. For example, any Σ such that

$$\int_{\Sigma} |\kappa(\Sigma)| d\Sigma < \infty, \quad \text{and} \quad \int_{\Sigma} \kappa(\Sigma) d\Sigma < -\varepsilon_0 < 0,$$

is parabolic. Therefore, choosing $R > 0$ sufficiently large, there exists a function φ such that

$$\begin{aligned} \varphi &\in C_0^\infty(\Sigma); \\ \varphi &\equiv 1 \text{ on } B(R); \\ 0 &\leq \varphi \leq 1; \\ \int_{\Sigma} |\nabla \varphi|^2 d\Sigma &< \frac{\varepsilon_0}{4}, \end{aligned}$$

and

$$\int_{B(R)} \kappa(\Sigma) d\Sigma < -\frac{1}{2}\varepsilon_0, \quad \int_{B(R)} \kappa(\Sigma) \varphi^2 d\Sigma < -\frac{1}{4}\varepsilon_0.$$

In this case,

$$\int_{\Sigma} (|\nabla \varphi|^2 + \kappa \varphi^2) d\Sigma < 0.$$

It follows that the surface is weakly κ parabolic. Now, since φ is compactly supported, there exists $R' > 0$ such that the support of φ is contained in $B(R')$. Then, it is always possible to change Σ outside of $B(R')$ such that the volume growth is of order R (for example, we can attach a cylinder $\partial B(R) \times \mathbb{R}^+$ to the compact manifold $B(R)$), which by Hartman's result [4] implies that for the new Σ' ,

$$\int_{\Sigma'} \kappa(\Sigma') d\Sigma' = 2\pi > 0.$$

Since

$$\Sigma \setminus B(R') \cong \Sigma' \setminus B(R'),$$

Σ' is still weakly κ parabolic, however the total Gauss curvature is positive.

4. PROOF OF THEOREM 1.4

We proceed as in the proof of Theorem 1.3 by demonstrating that

$$\sigma_0 < \sigma_{ess}.$$

By Proposition 2.1, this implies that the discrete spectrum is non-empty.

First, if the Gauss curvature is identically zero, then by [10, Theorem 2], the discrete spectrum is non-empty.

Henceforth, we shall assume that there is at least one point of Σ at which the Gauss curvature is positive. Then, by a theorem of Sacksteder [11, Theorem (*), page 610], with suitable choice of orientation, we can assume that the principle curvatures of Σ are always nonnegative.

By (1.5), it follows from the results of White [12] (Theorem 1, p. 318), that there exists an $\varepsilon_0 > 0$ such that for $R \gg 0$,

$$\int_{\partial B(R)} ||B|| > \varepsilon_0,$$

where B is the second fundamental form of Σ . Since the principle curvatures are nonnegative, we have

$$H \geq ||B||.$$

Thus we have

$$(4.1) \quad \int_{B(R_2) \setminus B(R_1)} H d\Sigma \geq \varepsilon_0 (R_2 - R_1)$$

provided that both R_1 and R_2 are large enough.

We follow the same general method of [2, 3, 8–10]. The main idea is to use the above estimate together with test functions *supported in annuli* whose radii tend to infinity.

Let $\varphi \in C_0^\infty(\Sigma \setminus B(\frac{R}{2}))$ be a smooth function such that

$$(4.2) \quad \begin{aligned} \text{supp}(\varphi) &\subset B(\frac{5}{2}R); \\ \varphi &\equiv 1 \text{ on } B(2R) \setminus B(R); \\ 0 &\leq \varphi \leq 1; \\ \int_{\Sigma} |\nabla \varphi|^2 d\Sigma &< \varepsilon_1, \end{aligned}$$

where $\varepsilon_1 \rightarrow 0$ as $R \rightarrow \infty$. The existence of such a function φ is guaranteed by the parabolicity of Σ .

Let χ be defined as in the previous sections. By Proposition 2.2, there is a constant c_2 such that

$$Q_1(\varphi\chi, \varphi\chi) \leq c_2 a \int_{\Sigma} |\nabla \varphi|^2 d\Sigma < c_2 a \varepsilon_1.$$

Hence

$$Q_1(\varphi\chi, \varphi\chi) \rightarrow 0 \text{ as } R \rightarrow \infty.$$

Next, we use the same calculations as in the preceding sections to compute

$$Q_2(\varphi\chi, \varphi\chi) = a \int_{\Sigma} \varphi^2 \kappa d\Sigma.$$

Since the support of φ is contained in the annulus $B(\frac{5}{2}R) \setminus B(R/2)$, by the integrability assumption (1.1) on κ ,

$$Q_2(\varphi\chi, \varphi\chi) \rightarrow 0 \text{ as } R \rightarrow \infty.$$

Since $Q = Q_1 + Q_2$, there exists $\varepsilon_3 > 0$ such that

$$Q(\varphi\chi, \varphi\chi) \leq \varepsilon_3, \quad \varepsilon_3 \rightarrow 0 \text{ as } R \rightarrow \infty.$$

Now let's consider a smooth function j on Σ with $0 \leq j \leq 1$, such that

$$\begin{aligned} j &\in C_0^\infty\left(B(\frac{5}{3}R) \setminus B(\frac{4}{3}R)\right); \\ j &\equiv 1 \text{ on } B(\frac{19}{12}R) \setminus B(\frac{17}{12}R); \\ |\nabla j| &< 2. \end{aligned}$$

We consider the function $j\chi(t)t$. By Lemma 2.1, the integrability of κ , and since $|\nabla j| < 2$, there is an absolute constant c_1 such that

$$Q(j\chi(t)t, j\chi(t)t) \leq c_1 a \int_{\Sigma} j^2 d\Sigma.$$

Next, let's consider $Q(\varphi\chi(t), j\chi(t)t)$. Since the support of j is contained in $\{\varphi \equiv 1\}$, by (2.6), $Q_1(\varphi\chi(t), j\chi(t)t) = 0$.

The same computation as in (3.4) shows that

$$Q_2(\varphi\chi(t), j\chi(t)t) = -\frac{a}{2} \int_{\Sigma} H j d\Sigma.$$

Let $\varepsilon > 0$. By our preceding calculations

$$(4.3) \quad \begin{aligned} & Q(\varphi\chi(t) + \varepsilon j\chi(t)t, \varphi\chi(t) + \varepsilon j\chi(t)t) \\ & < \varepsilon_3 - \varepsilon a \int_{\Sigma} H j d\Sigma + \varepsilon^2 c_1 a \int_{\Sigma} j^2 d\Sigma. \end{aligned}$$

By (4.1) and the definition of j , there is an independent constant c'_1 such that

$$Q(\varphi\chi(t) + \varepsilon j\chi(t)t, \varphi\chi(t) + \varepsilon j\chi(t)t) < \varepsilon_3 - \frac{a}{6} \varepsilon R + \varepsilon^2 c'_1 R^2.$$

Since $\varepsilon_3 \rightarrow 0$ as $R \rightarrow \infty$, we may first choose R sufficiently large and then choose $\varepsilon > 0$ appropriately so that for all $a \in (0, B_{\infty}^{-1})$,

$$Q(\varphi\chi(t) + \varepsilon j\chi(t)t, \varphi\chi(t) + \varepsilon j\chi(t)t) < 0.$$

It then follows from the definition of Q , the variational principle and Lemma 2.2 that

$$\sigma_0 < \sigma_{ess},$$

and therefore the discrete spectrum is non-empty. \square

5. PROOF OF THEOREM 1.5 AND FURTHER DISCUSSIONS

The proof of Theorem 1.4 can be generalized to demonstrate Theorem 1.5, which is a *stronger result* than Conjecture 1.

Proof. We first note that if the mean curvature vanishes identically, then since $\|B\| \rightarrow 0$ at infinity, in fact $B \equiv 0$ in this case. Therefore, $\kappa \equiv 0$, and the theorem follows from [2, 3]. So, we may assume

$$\int_{\Sigma} H^2 d\Sigma > 0.$$

Let $R > 0$ be sufficiently large such that

$$(5.1) \quad \int_{B(R)} H^2 d\Sigma > c > 0,$$

for some positive constant c . Using the capacity definition of parabolicity, for any $\varepsilon_1 > 0$, there exists a smooth function φ such that

$$\begin{aligned} \varphi &\in C_0^\infty(\Sigma); \\ \varphi &\equiv 1 \text{ on } B(2R); \\ 0 &\leq \varphi \leq 1; \\ \int_{\Sigma} |\nabla \varphi|^2 d\Sigma &< \varepsilon_1. \end{aligned}$$

First, we assume the mean curvature H is smooth. Then, define the function j so that

$$\begin{aligned} j &\in C_0^\infty(\Sigma); \\ j &\equiv H \text{ on } B(R); \\ j &\equiv 0 \text{ on } \Sigma \setminus B(2R); \\ \int_{\Sigma} H j d\Sigma &> \frac{c}{2}; \\ \int_{\Sigma} j^2 d\Sigma &< c_1; \\ \int_{\Sigma} |\nabla j|^2 d\Sigma &< c_2, \end{aligned}$$

for fixed constants c_1 and c_2 . Since j is supported on $\{\varphi \equiv 1\}$, the calculations in the preceding sections show that for any $\varepsilon > 0$,

$$\begin{aligned} Q(\varphi\chi + \varepsilon j\chi t, \varphi\chi + \varepsilon j\chi t) \\ \leq \varepsilon_1 - a\varepsilon \int_{\Sigma} H j d\Sigma + c_1 a \varepsilon^2 \int_{\Sigma} j^2 d\Sigma + c_2 a^3 \varepsilon^2 \int_{\Sigma} |\nabla j|^2 d\Sigma \end{aligned}$$

Choosing a sufficiently small, there is a universal constant c_3 such that

$$Q(\varphi\chi + \varepsilon j\chi t, \varphi\chi + \varepsilon j\chi t) \leq \varepsilon_1 - a\varepsilon \int_{\Sigma} H j d\Sigma + c_3 a \varepsilon^2 \int_{\Sigma} j^2 d\Sigma.$$

By definition of j and (5.1),

$$Q(\varphi\chi + \varepsilon j\chi t, \varphi\chi + \varepsilon j\chi t) \leq \varepsilon_1 - a\varepsilon \frac{c}{2} + c_3 a \varepsilon^2 c_1.$$

We may first let ε_1 tend to 0, and then choosing ε sufficiently small, it follows that

$$Q(\varphi\chi + \varepsilon j\chi t, \varphi\chi + \varepsilon j\chi t) < 0.$$

Finally, in the general case in which H need not be smooth, we use the mollification technique of [6] (see page 6, equation (2.5) and §4) and

define

$$H_a(x) := \frac{1}{\text{vol}(B_a(x))} \int_{B_a(x)} H d\Sigma,$$

where $B_a(x)$ is the geodesic ball of radius a about a point $x \in \Sigma$. In Russian literature, this is known as a “Steklov approximation.” Let j_a be a function defined analogously to the definition of j above, but with respect to the function H_a (instead of H). By Lemma 2.1, there exists a universal constant c such that

$$Q_1(j_a \chi t, j_a \chi t) \leq ca^3 \int_{\Sigma} |\nabla j_a|^2 d\Sigma.$$

By a straightforward calculation,

$$|\nabla H_a| = O(a^{-1}),$$

which implies the same estimate for $|\nabla j_a|$. Therefore, by Lemma 2.1, and since H_a converges pointwise to H as $a \rightarrow 0$, we may apply the above argument which shows that for all a sufficiently small, and all ε sufficiently small,

$$Q(\varphi \chi + \varepsilon j_a \chi t, \varphi \chi + \varepsilon j_a \chi t) < 0.$$

□

Remark 5.1. The preceding result implies Conjecture 1 and is in fact a stronger result, because not all parabolic surfaces have integrable Gauss curvature. For example, let $f \in \mathcal{C}^\infty(\mathbb{R})$ satisfy:

- (1) $\int_1^\infty f(r) dr = O(r^2)$;
- (2) $\int_1^\infty |f''(r)| dr = \infty$.

Let $\Sigma = \mathbb{R}^2$ with polar coordinates (r, θ) , and let the Riemannian metric on Σ be $g = dr^2 + f(r)^2 d\theta^2$. Then the volume growth is quadratic, and hence (Σ, g) is parabolic (see [7]), but by (2) above, the Gauss curvature is not integrable.

5.1. Further Discussions. In §7 of [3], they construct a layer whose discrete spectrum is empty, but this example is not asymptotically planar. In both our work and [2, 3], in some cases we are able to prove the existence of discrete spectrum for “thick layers,” meaning those whose thickness satisfies (1.6) and is not further restricted: Theorem 1.4; Theorem 1.1 (1), (3), (4); Theorem 1.2 (1), (3), (4). However, for the remaining cases, we must assume the layer is sufficiently thin to prove the existence of discrete spectrum. It would be interesting to investigate whether Theorem 1.5 holds for all “thick layers” over parabolic surfaces Σ satisfying (1.0.1), or whether one may construct a layer of

width $2a$ for a satisfying (1.6) over a parabolic surface which satisfies (1.0.1), such that the discrete spectrum of the layer is empty. The results of [2, 3] and Theorem 1.3 indicate that we expect the following conjecture holds.

Conjecture 2. *Let Σ be a surface which satisfies (1.0.1), and assume the Gauss curvature satisfies (1.1). Then, there exists $\alpha = \alpha(B_\infty) > 0$ depending only on the supremum of the norm of the second fundamental form such that for all $a \in (0, \alpha)$, the discrete spectrum of the quantum layer of width $2a$ over Σ is non-empty.*

Although we are unable to completely prove the above conjecture, the following results imply the conjecture holds under certain geometric assumptions.

Theorem 5.1. *Let Σ be a complete surface in \mathbb{R}^3 which satisfies the hypotheses (1.0.1), and assume that the Gauss curvature of Σ satisfies (1.1). The following is sufficient to imply Conjecture 2:*

$$(5.2) \quad \exists \varepsilon_0 > 0 \text{ such that } \forall R > 0, \exists j \in C_0^\infty(\Sigma \setminus B(R)) \text{ with}$$

$$\left(\int_\Sigma H j d\Sigma \right)^2 > \varepsilon_0 \int_\Sigma (|\nabla j|^2 + j^2) d\Sigma.$$

Proof. As noted previously, we may assume

$$\int_\Sigma \kappa d\Sigma > 0.$$

Since Σ is parabolic, for any $R > 0$ sufficiently large there exists a smooth function $\varphi \in C_0^\infty(B(R') \setminus B(R))$ such that

$$(5.3) \quad \begin{aligned} &\varphi \equiv 1 \text{ on } B(R'/2) \setminus B(2R); \\ &\int_\Sigma |\nabla \varphi|^2 d\Sigma < \varepsilon_1; \\ &0 \leq \varphi \leq 1; \\ &\int_\Sigma \kappa \varphi^2 d\Sigma < \varepsilon_1, \end{aligned}$$

where R' is a sufficiently large number. It follows from the arguments in the proof of Theorem 1.4 that

$$Q(\varphi\chi, \varphi\chi) < \varepsilon_3, \quad \varepsilon_3 \rightarrow 0 \text{ as } R \rightarrow \infty.$$

We choose R' large enough such that

$$\text{supp}(j) \subset B(R'/2) \setminus B(2R).$$

Without loss of generality, we assume that

$$\int_{\Sigma} H j d\Sigma > 0.$$

Then using the same method as in the proof of Theorem 1.4, letting $\varepsilon > 0$, we have

$$\begin{aligned} & Q(\varphi\chi + \varepsilon j\chi(t)t, \varphi\chi + \varepsilon j\chi(t)t) \\ & \leq \varepsilon_3 - a\varepsilon \int_{\Sigma} H j d\Sigma + c_1 \varepsilon^2 a \int_{\Sigma} (j^2 + |\nabla j|^2) d\Sigma \end{aligned}$$

for some constant c_1 which is *independent of R* . Since $\varepsilon_3 \rightarrow 0$ as $R \rightarrow \infty$, we first choose R sufficiently large to make ε_3 sufficiently small. Then, using (5.2), for all sufficiently small $\varepsilon > 0$,

$$Q(\varphi\chi + \varepsilon j\chi(t)t, \varphi\chi + \varepsilon j\chi(t)t) < 0.$$

This implies that the discrete spectrum of the layer $\Omega = \Sigma \times [-a, a]$ is non-empty, for any a which satisfies (1.6). \square

Based on the above theorem, we make the following *purely Riemannian geometric* conjecture.

Conjecture 3. *Let Σ be a complete surface in \mathbb{R}^3 which satisfies the hypotheses (1.0.1), and assume that the Gauss curvature κ of Σ satisfies (1.1). If the total Gauss curvature is positive*

$$\int_{\Sigma} \kappa > 0,$$

then (5.2) holds.

Remark 5.2. By the results of [2,3], Conjecture 3 together with Theorem 5.1 would imply Conjecture 2.

Remark 5.3. It is straightforward to verify that conditions (3) and (4) in Theorem 1.2 are each implied by (5.2). For example, to prove that (3) implies (5.2), j is replaced by jH , and (5.2) follows from a direct calculation. Assuming (4), (5.2) follows from either [3, Lemma 6.1] or [2, page 783].

The following proposition uses the result of White [12] to demonstrate a weaker version of the inequality of (5.2).

Proposition 5.1. *Assume that Σ is a complete surface in \mathbb{R}^3 which satisfies the hypotheses (1.0.1), and that the Gauss curvature κ satisfies*

$$\int_{\Sigma} |\kappa| d\Sigma < \infty, \quad \int_{\Sigma} \kappa d\Sigma > 0.$$

Then for any $R > 0$ there exists a positive constant ε_0 and a function $j \in C_0^\infty(\Sigma \setminus B(R))$ which satisfies

$$(5.4) \quad \left(\int_{\Sigma} j|H|d\Sigma \right)^2 > \varepsilon_0 \int_{\Sigma} (|\nabla j|^2 + j^2)d\Sigma.$$

Proof. By the result [12, Theorem 1, p. 318], for sufficiently large R ,

$$\int_{\partial B(R)} \|B\| > c_1 > 0$$

for some constant c_1 .

By the parabolicity of Σ and Proposition 3.1, we can find a function j whose support is contained in $B(\frac{5}{3}R) \setminus B(\frac{4}{3}R)$, with

$$(5.5) \quad \begin{aligned} j &\equiv 1 \text{ on } B(\frac{19}{12}R) \setminus B(\frac{17}{12}R); \\ \int_{\Sigma} |\nabla j|^2 d\Sigma &< 1, \\ \int_{\Sigma} j^2 d\Sigma &\leq c_1 R^2, \end{aligned}$$

where c_1 is a positive constant independent of R . Hence we have

$$\int_{\Sigma} \|B\| j d\Sigma \geq c_2 R$$

for some constant c_2 .

On the other hand, since κ is integrable, for sufficiently large R , we have

$$\int_{\Sigma} j \sqrt{|\kappa|} d\Sigma \leq \sqrt{\int_{\Sigma} j d\Sigma} \cdot \sqrt{\int_{\Sigma} j \kappa d\Sigma} \leq \varepsilon_3 R,$$

for some small positive constant ε_3 when R is large.

Since $|H| \geq \|B\| - \sqrt{2|\kappa|}$, the above inequalities show that

$$\int_{\Sigma} j|H|d\Sigma \geq \int_{\Sigma} \|B\| j d\Sigma - \int_{\Sigma} j \sqrt{2|\kappa|} d\Sigma \geq (c_2 - \sqrt{2}\varepsilon_3)R.$$

Therefore, there exists $\varepsilon > 0$ such that for sufficiently large R ,

$$\int_{\Sigma} j|H|d\Sigma > \varepsilon R.$$

Finally, by definition of j , for sufficiently large R ,

$$\int_{\Sigma} (|\nabla j|^2 + j^2) d\Sigma < 2c_1 R^2.$$

Therefore, there exists a constant $\varepsilon_0 > 0$ such that for all R sufficiently large,

$$\left(\int_{\Sigma} j|H|d\Sigma \right)^2 > \varepsilon_0 \int_{\Sigma} (|\nabla j|^2 + j^2)d\Sigma.$$

□

Remark 5.4. Note that the estimate (5.4) is weaker than (5.2), because the integration is $j|H|$ rather than jH . Although we are unable with our present methods to prove Conjecture 3, the proposition supports the conjecture since it shows that if the mean curvature has fixed sign off some compact set, then Conjecture 3 holds. However, by our methods, we cannot prove the conjecture when the mean curvature H continually oscillates between positive and negative all the way to infinity; c.f. Example 6 from §6 of [3]. One would need a new and different argument to prove Conjecture 3 in such cases.

Our final theorem below shows that if the surface Σ satisfies certain isoperimetric inequalities, this is sufficient for (5.2).

Theorem 5.2. *Let Σ satisfy the hypotheses (1.0.1), and assume Σ also satisfies the following.*

- (1) *The isoperimetric inequality holds. That is, there is a positive constant δ_1 such that if D is a domain in Σ , we have*

$$(\text{length}(\partial D))^2 \geq \delta_1 \text{Area}(D).$$

- (2) *There is another positive constant δ_2 such that for any compact set $K \subset \Sigma$, there is a curve $C \subset \Sigma \setminus K$ such that if $\vec{\gamma}$ is the normal vector of C in Σ , then there is a vector $\vec{\eta}$ in \mathbb{R}^3 such that*

$$\langle \vec{\gamma}, \vec{\eta} \rangle \geq \delta_2 > 0.$$

- (3) *All such curves C are tamed. That is, let $\sigma(t, x)$ be the geodesic flow of $\vec{\gamma}$. Then there exist constants δ_3, A such that the following hold.*

- (3.1) $\sigma(t, x)$ is defined up to $|t| < \delta_3$;
- (3.2) the map $C \times (-\delta_3, \delta_3) \rightarrow \Sigma$ is diffeomorphic onto its image;
- (3.3) the derivatives of σ and its inverse are bounded by the fixed constant A .

Then, if the Gauss curvature satisfies (1.1), (5.2) is valid.

Proof. To construct the required function j satisfying (5.2), we let ρ be a smooth non-increasing cut-off function such that

$$\rho(t) = \begin{cases} 1, & t \leq 1; \\ 0, & t \geq 1. \end{cases}$$

Let C be the curve outside a compact set K , and let D be the compact domain of Σ such that $\partial D = C$. Assume that $\vec{\gamma}$ is the outward norm of C in Σ .

Define the cut-off function $\tilde{\rho}$ on Σ as follows

$$\tilde{\rho} = \begin{cases} 1 & x \in D \\ \rho(\varepsilon^{-1} \text{dist}(x, C)) & x \notin D \end{cases},$$

where ε is a positive number to be determined later.

Let (x, y, z) be the standard coordinates of \mathbb{R}^3 . Without loss of generality, assume the vector $\vec{\eta}$ in the hypotheses of the theorem is the z -direction in three-dimensional Euclidean space.

Let \vec{n} be the normal vector of $\Sigma \subset \mathbb{R}^3$. Let n_z be the z -component of \vec{n} . Define the function j by

$$j = \tilde{\rho} \cdot n_z.$$

Since $|\rho n_z| + |\nabla(\rho n_z)| \leq \|B\|_\infty + 1$, if we choose ε small enough, we have

$$\int_{\Sigma} (|\nabla j|^2 + j^2) d\Sigma \leq C \varepsilon^{-2} \text{Area}(D)$$

for ε small, where the constant C depends on A .

On the other hand, since $Hn_z = \Delta z$, we have

$$\begin{aligned} \left| \int_{\Sigma} H \tilde{\rho} n_z d\Sigma \right| &= \left| \int_{\Sigma} \nabla z \nabla \tilde{\rho} d\Sigma \right| \\ &= (\langle \vec{\gamma}, \nabla z \rangle + o(1)) \cdot \text{length}(C) > \frac{1}{2} \delta_2 \cdot \text{length}(C). \end{aligned}$$

Therefore using the isoperimetric inequality, the conclusion of the theorem holds. □

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